Parity-dependent potentials for the one-dimensional Schrodinger equation obtained from inverse spectral theory

1983 J. Phys. A: Math. Gen. 16 303

(http://iopscience.iop.org/0305-4470/16/2/013)

J. Phys. A: Math. Gen. **16** (1983) 303–316. Printed in Great Britain

Parity-dependent potentials for the one-dimensional Schrödinger equation obtained from inverse spectral theory

P B Abraham†, B DeFacio‡§ and H E Moses‖

† Naval Underwater Systems Center, New London, Connecticut 06385, USA
‡ Ames Laboratory-USDOE and Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA
‖ University of Lowell, Center for Atmospheric Research, Lowell, Massachusetts 01854, USA

Received 12 March 1982, in final form 11 August 1982

Abstract. The present paper arises from the examination of how the triangularity conditions on the Gelfand–Levitan kernel affect the nature of the potentials. The original triangularisation property for the one-dimensional problem led to local potentials. The triangularity conditions used in the present paper (which is the one-dimensional analogue of the three-dimensional conditions used in a previous paper) lead to simple non-local potentials, which because of their form we call parity-dependent potentials. The inverse spectral theory problem is solved explicitly for several types of spectral measure functions. Such solutions give parity-dependent potentials with complete sets of eigenfunctions in terms of elementary functions. Using these examples, it is shown that a rich spectral theory exists with some aspects strikingly different from those of the spectral theory for local potentials.

1. Introduction. Parity-dependent potentials

The object of the present paper is to introduce a very simple class of non-local potentials for the one-dimensional Schrödinger equation ($-\infty < x < \infty$). These non-local potentials depend on the parity operator as well as the x coordinate and are simpler than the sometimes-used momentum-dependent potentials. Explicit examples of the non-local potentials are given for which the Schrödinger equation can be solved for exactly, and for which complete sets of eigenfunctions can be found. Hamiltonians with these non-local potentials have unusual spectral properties compared with those arising from those having local potentials, and therefore the new potentials enrich our ideas about the possibilities for the spectrum of more general operators. Examples of the new properties are that point eigenvalues can be embedded in the continuum and that a point eigenvalue can have the multiplicity 2. Also non-local potentials can be given which do not scatter at all for any energy of incoming wave.

We came across this new class of potentials as a consequence of studying the inverse spectral theory for the one-dimensional Schrödinger equation. In fact, the

§ Permanent address: Department of Physics, University of Missouri-Columbia, Columbia, Missouri 65211, USA.

0305-4470/83/020303+14$02.25

specific examples of non-local potentials given in the present paper were obtained using inverse spectral theory. There is a large literature for inverse spectral theory (see e.g. Kay and Moses 1955, 1956a, b, Gelfand and Levitan 1951, Abraham and Moses 1980). For the proper understanding of how the non-local potentials make their appearance as motivated in the present paper, there should be some familiarity with the inverse spectral theory background referred to above. However, the properties of the examples given in the present paper can be verified independently of the inverse theory. Nevertheless, the way the new non-local potentials came to our attention was through inverse spectral theory and we must refer to it, even though perfunctorily. We proceed to summarise some aspects of the direct and inverse spectral problems

In the direct spectral problem for Hermitian operators, one considers a Hamiltonian as given and seeks a complete set of eigenfunctions, of both the discrete and continuous spectrum. The completeness of the set is stated in terms of the spectral measure function, which tells one what the nature of the spectrum is; also its functional form, for given boundary conditions on the eigenfunctions, is dependent upon the potential. In some cases which we shall refer to shortly, the boundary conditions can be picked so that the spectral measure function is essentially a portion of the scattering operator and the normalisations of the eigenfunctions for the point eigenvalues.

To summarise, the direct spectral problem is the following: given the potential, and boundary conditions on the eigenfunctions, find the spectral measure function (which gives the resolution of the identity or, equivalently, the completeness relation for the eigenfunctions).

The inverse spectral problem interchanges some of the information which is given in the direct spectral problem with that which is to be found. The version which this paper deals with is the following: given boundary conditions on the eigenfunctions and the spectral measure function compatible with the boundary conditions, find the potential.

The direct and inverse problems are discussed in a very general manner in Kay and Moses (1955, 1956a, b) and the one-dimensional Schrödinger equation is treated in great detail. The method used to set up and solve the inverse problem for the one-dimensional case is a generalisation of that of Gelfand and Levitan (1951) for the radial equation. One uses the Gelfand–Levitan equation, which is a linear integral equation for a kernel (the Gelfand–Levitan kernel) which may be considered the kernel of a linear operator. The input of the linear integral equation is the spectral measure function. The kernel allows one to compute the potential.

In Kay and Moses (1955, 1956a, b) it is shown that to solve the general Gelfand–Levitan equation a triangularity condition must be imposed upon the kernel which then becomes a Volterra kernel. A way is given for choosing the triangularity condition which implies boundary conditions on the eigenfunctions which is one of the given pieces of information in the inverse spectral problem. In the one-dimensional case this corresponds to choosing the eigenfunctions to be the well known Jost wavefunctions. This choice of triangularity also enables one to obtain the potential, which is a local one. Many explicit examples of the use of the one-dimensional inverse spectral theory are given in Kay and Moses (1955, 1956a, b), Gelfand and Levitan (1951), Abraham and Moses (1980), for example. The choice of boundary conditions for the one-dimensional problem has the fortuitous property that the spectral measure function is very simply related to the reflection coefficient of one-dimensional scattering. Therefore one can speak of the inverse *scattering* problem in one dimension.

In this paper we study the effect of the choice of a different triangularisation condition for the Gelfand–Levitan kernel on the boundary conditions of the eigenfunctions and on the nature of the potential in one dimension. We shall see that we are led to a class of simple non-local potentials. Moreover, we can give explicit examples of the construction of these potentials from given spectral measures using inverse methods. The potentials will be the parity-dependent potentials of this paper.

It should be mentioned that our researches on the consequences of the triangularity condition of the Gelfand–Levitan kernel actually began for the three-dimensional Schrödinger equation, where it was shown that explicit non-local potentials can be found for which the Hamiltonian has complete sets of eigenfunctions (Moses 1979), and that the present paper represents an adaptation of the one-dimensional problem, for which the mathematics is simpler. (In Kay and Moses (1961a, b), Faddeev (1976) and Newton (1974) other triangularity conditions are used for the three-dimensional problem, which could lead to local potentials, but no concrete examples are given.) It should also be mentioned that in Newton (1980) a very similar triangularisation condition to ours is given for the one-dimensional problem, but the author confines his interest to local potentials and thus non-local potentials of the present paper are not found.

We define an operator V to be a parity-dependent potential if it acts on an element $\Psi(x)$ of the Hilbert space in the following way:

$$V\Psi(x) = V_1(x)\Psi(x) + V_2(x)\Psi(-x). \tag{1}$$

The operator V is thus given when the functions $V_1(x)$, $V_2(x)$ are given. Equation (1) can also be written

$$V\Psi(x) = V_1(x)\Psi(x) + V_2(x)P\Psi(x), \tag{1a}$$

where P is the parity, defined as usual by

$$P\Psi(x) = \Psi(-x). \tag{1b}$$

Because of equation (1) the Schrödinger equation gives a dependence on the past and future as well as the present.

If the parity-dependent potential V is Hermitian, $V_1(x)$ is a real function of x, and $V_2^*(-x) = V_2(x)$. The asterisk means complex conjugate. The Hermiticity of the parity-dependent potential V is readily shown. Let $\Phi(x)$ and $\Psi(x)$ be two state vectors. Then

$$\begin{aligned}(\Phi, V\Psi) &= \int_{-\infty}^{+\infty} \Phi^*(x) V\Psi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \Phi^*(x)[V_1(x)\Psi(x) + V_2(x)\Psi(-x)]\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \Psi(x) V_1(x)\Phi^*(x)\,\mathrm{d}x + \int_{-\infty}^{+\infty} \Psi(x) V_2(-x)\Phi^*(-x)\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \Psi(x) V_1(x)\Phi^*(x)\,\mathrm{d}x + \int_{-\infty}^{+\infty} \Psi(x) V_2^*(x)\Phi^*(-x)\,\mathrm{d}x \\ &= [(\Psi, V\Phi)]^*.\end{aligned}$$

In a certain heuristic sense, the operator V is the most general 'function' of P and x which one can have. Consider

$$V = V(x, P) \tag{2}$$

and formally expand this operator in a power series in P. Since P does not commute with x, a typical term would have powers of P distributed among functions of x, i.e. any of the nth-order terms would have the form

$$P^{j}V_1P^{j}V_2 \ldots V_sP^{r}$$

where the sum of exponents on the parity operator P is n, and V_j are functions of x.

However, $P^j = I$ if j is even and $P^j = P$ if j is odd. Moreover, $PV_k(x) = V_k(-x)P$. It then follows that any of the nth terms of the expansion can always be written either as $V_e(x)$ if n is even or as $V_o(x)P$ if n is odd (the subscripts 'e' and 'o' refer to even or odd). Summing all terms of the expansion leads to

$$V = V_1(x) + V_2(x)P. \tag{1c}$$

2. The Gelfand–Levitan equation. The potential in terms of the kernel

The techniques of Kay and Moses (1955, 1956a, b) can be applied immediately, once one has specified the triangularity of the Gelfand–Levitan kernel. For the sake of terseness we shall not repeat the arguments, but use the results of Kay and Moses (1955, 1956a, b) with slight changes in notation.

As usual we take $(-\infty < x < \infty)$ and define H_0 by

$$H_0 = -\mathrm{d}^2/\mathrm{d}x^2. \tag{3}$$

The operator A_0 has the eigenvalues ± 1 which correspond to the directions of the momentum as in Kay and Moses (1955, 1956a, b); $|H_0, A_0; E, a\rangle$ is a simultaneous eigenstate of H_0, A_0 with eigenvalues E, a respectively. The eigenfunctions $\langle x|H_0, A_0; E, a\rangle$ are given by

$$\langle x|H_0, A_0; E, a\rangle = (2\pi^{1/2}E^{1/4})^{-1} \exp(\mathrm{i}aE^{1/2}x) \qquad (0 < e < \infty, a = \pm 1). \tag{4}$$

They satisfy the orthogonality and completeness relations (2.3) and (2.4) of Kay and Moses (1956 b).

As in Kay and Moses (1955, 1956a, b) we introduce the continuous spectrum part of the weight operator $\langle a|\omega_c(E)|a'\rangle$ which is a function of E, a, a' and satisfies the Hermiticity condition

$$\langle a|\omega_c(E)|a'\rangle = [\langle a'|\omega_c(E)|a\rangle]^*. \tag{5}$$

In the case of the local potentials of Kay and Moses (1955, 1956a, b) the continuous part of the weight operator is related to the scattering operator. In the present case, because of the triangularity condition which we wish to impose on $\langle x|K|x'\rangle$, where $\langle x|K|x'\rangle$ is the Gelfand–Levitan kernel, this relation is quite complicated and we take the weight operator as arbitrary, except that we require it to be a positive-definite matrix in the variables a, a' for every E.

Thus the continuous part of the weight operator W_c is given by

$$\langle x|W_c|x'\rangle = \sum_a \sum_{a'} \int_0^\infty \langle x|H_0, A_0; E, a\rangle\langle a|\omega_c(E)|a'\rangle\langle H_0, A_0; E, a'|x'\rangle \,\mathrm{d}E. \tag{6}$$

Equation (6) takes on a pleasanter form if we define (keeping the Hermiticity of $\langle a|\omega_c(E)|a'\rangle$ in mind)

$$\begin{aligned} &c(k)=\langle 1|\omega_c(E)|1\rangle-1, \qquad d(k)=\langle -1|\omega_c(E)|-1\rangle-1, \\ &b(k)=\langle -1|\omega_c(e)|+1\rangle, \qquad b^*(k)=\langle +1|\omega_c(E)|-1\rangle \qquad (k=E^{1/2}, c(k), d(k) \text{ real}). \end{aligned} \tag{7}$$

The quantities $c(k)$, $d(k)$, $b(k)$ have been defined for $k>0$. It is also useful to introduce $k<0$, by defining

$$\begin{aligned} &c(-k)=d(k), \qquad c(k) \text{ real, all } k, \\ &b(-k)=b^*(k). \end{aligned} \tag{7a}$$

Then

$$\begin{aligned} &\langle x|W_c|x'\rangle=\delta(x-x')+F(x+x')+G(x-x'), \\ &G(x)=(2\pi)^{-1}\int_{-\infty}^{+\infty} c(k)\,e^{ikx}\,dk, \\ &F(x)=(2\pi)^{-1}\int_{-\infty}^{+\infty} b(k)\,e^{-ikx}\,dk. \end{aligned} \tag{8}$$

The discrete portion of the weight operator is given by

$$\langle x|W_d|x'\rangle=\sum_i \frac{\Psi_{0i}(x)\Psi_{0i}^*(x')}{C_i}, \tag{9}$$

where $\Psi_{0i}(x)$ are the formal eigenfunctions of H_0 which satisfy

$$H_0\Psi_{0i}(x)=E_i\Psi_{0i}(x). \tag{10}$$

The eigenvalues E_i are prescribed and are to be the eigenvalues of H. They can take on any real value. The fact that for our new triangularity condition equation (10) admits two linearly independent solutions for a given E_i leads to the possibility of having a multiplicity of 2 for E_i, if one chooses. The positive constants C_i will be the normalisations of the corresponding eigenfunctions of H.

We now come to the triangularity condition which we shall impose:

$$\langle x|K|x'\rangle=\langle x|K_0|x'\rangle=0 \qquad \text{for } |x'|>|x|. \tag{11}$$

The Gelfand–Levitan equation is then

$$\begin{aligned} \langle x|K|x'\rangle=&-F(x+x')-G(x-x')-\langle x|W_d|x'\rangle \\ &-\int_{-|x|}^{|x|}\langle x|K|x''\rangle F(x''+x')\,dx'' \\ &-\int_{-|x|}^{|x|}\langle x|K|x''\rangle G(x''-x')\,dx'' \\ &-\int_{-|x|}^{|x|}\langle x|K|x''\rangle\langle x''|W_d|x'\rangle\,dx'' \qquad (|x'|\leq|x|). \end{aligned} \tag{12}$$

The triangularity condition (11) determines the form of the potential.

In the appendix it is proved that

$$\langle x|V|x'\rangle = 4\delta(x)\delta(x')\langle 0|K|0\rangle + \operatorname{sgn} x\,\delta(x-x')2\frac{\mathrm{d}}{\mathrm{d}x}\langle x|K|x\rangle$$

$$+\operatorname{sgn} x\,\delta(x+x')2\frac{\mathrm{d}}{\mathrm{d}x}\langle x|K|-x\rangle, \tag{13}$$

so that, writing an arbitrary state $\langle x|\psi\rangle \equiv \psi(x)$, we have that $V\psi(x)$ is given by (1) with

$$V_1(x) = 4\delta(x)\langle 0|K|0\rangle + 2\operatorname{sgn} x\frac{\mathrm{d}}{\mathrm{d}x}\langle x|K|x\rangle, \qquad V_2(x) = 2\operatorname{sgn} x\frac{\mathrm{d}}{\mathrm{d}x}\langle x|K|-x\rangle. \tag{14}$$

Denoting the continuous spectrum eigenfunctions of H by $\Psi_k(x)$ (we are using a notation close to that of Kay and Moses (1956b)) where

$$\Psi_k(x) = \mathrm{e}^{\mathrm{i}kx} + \int_{-|x|}^{x|} \langle x|K|x'\rangle \mathrm{e}^{\mathrm{i}kx'}\,\mathrm{d}x', \tag{15}$$

$$H\Psi_k(x) = k^2\Psi_k(x). \tag{16}$$

The eigenfunctions corresponding to the point eigenvalue E_i are given by

$$\Psi_i(x) = \Psi_{0i}(x) + \int_{-|x|}^{|x|} \langle x|K|x'\rangle\Psi_{0i}(x')\,\mathrm{d}x' \tag{17}$$

and satisfy

$$H\Psi_i(x) = E_i\Psi_i(x). \tag{18}$$

The eigenfunctions satisfy the completeness relation

$$\frac{1}{2\pi}\left(\int_{-\infty}^{+\infty}\Psi_k(x)[1+c(k)]\Psi_k^*(x')\,\mathrm{d}k + \int_{-\infty}^{+\infty}\Psi_k(x)b(-k)\Psi_{-k}^*(x')\,\mathrm{d}x'\right)$$

$$+\sum_i \frac{\Psi_i(x)\Psi_i^*(x')}{C_i} = \delta(x-x'). \tag{19}$$

3. Local potentials

An important question in the use of the algorithm is the question of whether one can obtain a Gelfand–Levitan kernel $\langle x|K|x'\rangle$ with the appropriate triangularity condition for all parity-dependent potentials. One way to prove this statement—if indeed it is true—is to find an appropriate Green function and obtain an integral equation for the kernel (without initially assuming the triangularity property) and obtain an integral equation for the kernel in terms of the potential and the Green function. The integral equation would then give the triangularity condition on the kernel and show one how the potential is to be obtained from the kernel. This program is carried out for the usual triangularity condition $\langle x|K|x'\rangle \equiv 0$ for $x > x'$ in Moses (1975).

However, we have been unable to find an appropriate Green function thus far for the present triangularisation condition. But for the special case that the potential is local and there is no delta function contribution at the origin, we have found a Green

function from which we can construct a Gelfand–Levitan kernel with the right properties. The Green function is

$$G_k(x, x') = [\eta(x')\eta(x - x') - \eta(-x')\eta(x' - x)](\sin k(x - x'))/k \tag{20}$$

where $\eta(x)$ is the Heaviside function $\eta(x) = 0$ for $x < 0$, $\eta(x) = 1$ for $x > 0$. We shall not go through the calculations which lead to the integral equation for the Gelfand–Levitan kernel, but merely state that it parallels the treatment of Abraham *et al* (1982) very closely. In addition to obtaining the right triangularity conditions on the kernel, and the first of equations (14) for $V_1(x)$ (without the δ function at the origin), it is shown that

$$\langle x|K|-x\rangle \equiv 0. \tag{21}$$

It follows from (14) that (21) is a necessary and sufficient condition for the potential to be local without a δ function at the origin.

It would be nice to find a condition on $\langle x|\Omega|x'\rangle = F(x + x') + G(x - x') + \langle x|W_d|x'\rangle$ to assure us that (21) is valid and that the potential is local. We have not yet been able to give a general condition on the spectral function. However, an important special case can be considered.

Let $c(k) \equiv 0$. Moreover choose $b(k)$ to have the properties of the reflection coefficient of the usual one-dimensional scattering problem (Kay and Moses 1956b). Furthermore, let the point eigenvalues E_i be negative,

$$E_i = -\kappa_i^2, \tag{22}$$

and choose

$$\Psi_{i0}(x) = \exp(\kappa_i x). \tag{23}$$

Finally, choose $b(k)$, κ_i, and C_i so that

$$\langle x|\Omega|0\rangle \equiv 0 \qquad \text{for } x < 2\alpha \text{ with } \alpha > 0. \tag{24}$$

(Conditions which lead to equation (24) are discussed in Kay and Moses (1956b).) Then the Gelfand–Levitan equation of the present paper is identical to the equation of Kay and Moses (1956b) for the usual one-dimensional inverse problem for potentials $V_1(x) \equiv V(x)$ such that $V(x) \equiv 0$ for $x < -\alpha$. The quantity $b(k)$ is the reflection coefficient. Thus the Gelfand–Levitan equation of the present paper leads to local potentials under at least some conditions on the spectral measure function. It should also be noted that equation (21) holds under the conditions we are discussing at present. (Of course a similar statement can be made for $x > 2\alpha$ $(\alpha > 0)$ using the Gelfand–Levitan equation for local potentials 'from the other side.')

4. Examples of parity-dependent potentials

It is important to provide concrete examples of parity-dependent potentials to make sure that the formalism is not empty and to show that such potentials have an interesting spectral theory. All our examples are cases in which $b(k) \equiv c(k) \equiv 0$. The spectral measure functions are constructed entirely from point eigenvalue information and the potentials are thus analogues of the reflectionless potentials of Kay and Moses (1956c).

4.1.

Assume a single point eigenvalue $E_1 = 0$. We have two linearly independent choices for $\Psi_{01}(x)$. as the simplest possible case we take

$$\Psi_{01}(x) = 1. \tag{25}$$

We write $C_1 = C$.

Even with this simple choice for the spectral data we obtain interesting results. The Gelfand–Levitan equation is

$$\langle x|K|x'\rangle = -C^{-1} - C^{-1}\int_{-|x|}^{|x|} \langle x|K|x'\rangle \, dx'. \tag{26}$$

We note that $\langle x|K|x'\rangle$ is independent of x'. Thus we write

$$\langle x|K|x'\rangle = F(x). \tag{27}$$

Thus equation (30) becomes

$$F(x) = -C^{-1} - C^{-1}2|x|F(x). \tag{28}$$

The function $F(x)$ is easily solved for and we have

$$\langle x|K|x'\rangle = -(2|x| + C)^{-1}, \tag{29}$$

$$V_1(x) = -(4/C)\delta(x) + 4/(2|x| + C)^2, \tag{30}$$

$$V_2 = 4/(2|x| + C)^2. \tag{31}$$

The eigenfunction corresponding to the point eigenvalue 0 is

$$\Psi_1(x) = C/(2|x| + C) \tag{32}$$

and the eigenfunctions of the continuous spectrum are

$$\Psi_k(x) = e^{ikx} - (2 \sin k|x|)/k(2|x| + C). \tag{33}$$

That our solutions satisfy the Schrödinger equation with the potential given above and that they satisfy the completeness relation is verified in a straightforward but not trivial way.

Usually the eigenfunctions $\Psi_k(x)$ of the present paper are not the outgoing wavefunctions. It will be remembered that the outgoing wavefunctions are defined by boundary conditions. One half satisfy

$$\lim_{x \to -\infty} \Psi_k^{\text{out}}(x) = e^{ikx} + b(k)\, e^{-ikx},$$

$$\lim_{x \to +\infty} \Psi_k^{\text{out}}(x) = t(k)\, e^{ikx} \qquad \text{for } k > 0, \tag{34}$$

and the other half satisfy

$$\lim_{x \to +\infty} \Psi_k^{\text{out}}(x) = e^{-ikx} + c(k)\, e^{ikx},$$

$$\lim_{x \to -\infty} \Psi_k^{\text{out}}(x) = s(k)\, e^{-ikx} \qquad \text{for } k > 0. \tag{35}$$

The sum of the sets is a complete set if there are no point eigenvalues. The quantities $b(k)$ and $t(k)$ are the reflection and transmission coefficients from the left and $c(k)$ and $s(k)$ are the corresponding quantities from the right. These quantities describe the scattering properties of interactions whether or not they are local potentials.

For the potential of equation (31) we see that $\Psi_k(x)$, $\Psi_k^*(x)$ given by equation (33) satisfies the outgoing boundary conditions (34) and (35) respectively, but with all the transmission and reflection coefficients equal to zero. Thus there are parity-dependent potentials which do not scatter. There is no scattering experiment which would reveal the existence of these potentials.

4.2.

As a second example of a parity-dependent potential which comes from our Gelfand–Levitan equation, we again take the case of a single point eigenvalue $E_1 = 0$ and a normalisation $C_1 = C$. But instead of selecting the function $\Psi_{01}(x)$ as given by equation (26), we take

$$\Psi_{01}(x) = x. \tag{36}$$

On substituting in the Gelfand–Levitan equation we see that $\langle x|K|x'\rangle$ has the form

$$\langle x|K|x'\rangle = F(x)x'. \tag{37}$$

On substituting this form back into the Gelfand–Levitan equation, we obtain an equation for $F(x)$ which is easily solved for, and we find

$$\langle x|K|x'\rangle = -\tfrac{3}{2}xx'/(|x|^3 + \tfrac{3}{2}C). \tag{38}$$

Thus

$$V_1(x) = -V_2(x) = 3|x|(|x|^3 - 3C)/(|x|^3 + \tfrac{3}{2}C)^2. \tag{39}$$

The eigenfunction corresponding to the point eigenvalue $E_1 = 0$ is

$$\Psi_1(x) = \tfrac{3}{2}Cx/(|x|^3 + \tfrac{3}{2}C) \tag{40}$$

and those for the continuous spectrum are

$$\Psi_k(x) = e^{ikx} - [3i|x|^3/(|x|^3 + \tfrac{3}{2}C)]j_1(kx). \tag{41}$$

In equation (44) j_1 is the usual spherical Bessel function of order 1. Again this potential does not scatter and the outgoing eigenfunctions are $\Psi_k(x)$, $\Psi_k^*(x)$.

In these two examples it is surprisingly difficult to verify our results using direct spectral theory methods. Yet from an inverse spectral theory point of view they come easily from the Gelfand–Levitan equation.

4.3.

As a third example of the unexpected behaviour of parity-dependent potentials, we consider the case of a point eigenvalue of multiplicity 2 embedded in the continuous spectrum. Accordingly, let $E_1 = \lambda^2 = e_2$ with $\lambda > 0$ be the positive point eigenvalue and $c_1 = C$ the normalisation for both the eigenfunctions. One may choose any two linearly independent solutions of equation (10), and for this example we pick

$$\Psi_{01} = e^{i\lambda x}, \qquad \Psi_{02} = e^{-i\lambda x} \tag{42}$$

and thus

$$\langle x|W_d|x'\rangle = (2/C)\cos\lambda(x-x') = C^{-1}(e^{i\lambda(x-x')} + e^{-i\lambda(x-x')}). \tag{43}$$

As in the above examples, this kernel is separable, and from the Gelfand–Levitan equation it is seen that $\langle x|K|x'\rangle$ has the form

$$\langle x|K|x'\rangle = F_1(x)e^{i\lambda x} + F_2(x)\,e^{-i\lambda x'}. \tag{44}$$

On substituting into the Gelfand–Levitan equation, one readily obtains F_1 and F_2. The final answer for the Gelfand–Levitan kernel is

$$\langle x|K|x'\rangle = \frac{2[C\cos\lambda(x+x')\sin\lambda|x| - \cos\lambda(x-x')(C+2|x|)]}{C^2\cos^2\lambda x + 4|x|C + 4x^2}. \tag{45}$$

One can readily obtain the eigenfunctions $\Psi_k(x)$, $\Psi_1(x)$, $\Psi_2(x)$ and the potentials $V_1(x)$ and $V_2(x)$. Since the results are rather long expressions involving elementary functions but are otherwise quite normal-looking, we shall not write these quantities explicitly. This potential does not scatter either.

4.4.

As a final example we consider the case where there is one negative point eigenvalue $E_1 = -\kappa^2$. For $\Psi_{01}(x)$ and C_1 we take

$$\Psi_{01}(x) = e^{\kappa x}, \qquad C_1 = C. \tag{46}$$

The Gelfand–Levitan equation is easily solved for. We find

$$\langle x|K|x'\rangle = -e^{\kappa(x+x')}/(\kappa C + \sinh 2\kappa|x|). \tag{47}$$

Furthermore,

$$\begin{aligned} V_1(x) &= -\frac{4\kappa^2 e^{2\kappa x}}{(\kappa C + \sinh 2\kappa|x|)^2}(\kappa C\ \mathrm{sgn}\ x - e^{-2\kappa x}) - \frac{4}{C}\delta(x), \\ V_2(x) &= (4\kappa^2\cosh 2\kappa x)/(\kappa C + \sinh 2\kappa|x|)^2. \end{aligned} \tag{48}$$

Also the eigenfunctions of the continuous spectrum are

$$\Psi_k(x) = e^{ikx} - \frac{2\kappa\, e^{\kappa x}}{k - i\kappa}\,\frac{\sin[(k-i\kappa)|x|]}{\kappa C + \sinh 2\kappa|x|}, \tag{49}$$

$$\Psi_1(x) = \kappa C\, e^{\kappa x}/(\kappa C + \sinh 2\kappa|x|). \tag{50}$$

Unlike the previous examples, this potential scatters. The outgoing eigenfunctions are related to $\Psi_k(x)$ by

$$\Psi_k^{\mathrm{out}}(x) = \Psi_k(x), \tag{51}$$

for the outgoing functions satisfying the boundary conditions equation (37), and

$$\Psi_k^{\mathrm{out}}(x) = [(k+i\kappa)/(k-i\kappa)]\Psi_k^*(x) \tag{52}$$

for those satisfying the boundary condition equation (38). It is readily seen that

$$b(k) = c(k) = 0 \qquad t(k) = s(k) = (k+i\kappa)/(k-i\kappa). \tag{53}$$

The scattering operator for this parity-dependent potential is identical to that for the simplest of the reflectionless potentials of Kay and Moses (1956c). Hence we have

shown that knowledge of the scattering operator and position of the point eigenvalues is not sufficient to determine whether a potential is non-local or not. This point is discussed more fully in Abraham *et al* (1982).

Acknowledgments

The work of BDF was supported by the US Department of Energy, Contract No W-7405-eng-82, Division of Engineering, Mathematical and Geosciences, budget code AK-01-04-02, and the Institute for Theoretical Physics, Chalmers Technological University, Göteborg, Sweden. BDF also thanks the University of Missouri-Columbia Physics Department for a leave of absence 1979–81 and the Institute for Theoretical Physics, Chalmers Technological University, S412-96, Göteborg, Sweden, for a pleasant and productive visit during November 1980 when part of this work was performed. A preliminary report was given at the meeting RCP 264, *Recontre Inderdisciplinaire Problems Inverse*, December 3, 1980, Monpellier, France, using parts of this paper.

The work of HEM was supported by the Army Research Office under Grant No DAAG29-78-G-0003 and the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No AFOSR 81-0253.

Appendix 1. Derivation for the non-local potential in terms of the Gelfand–Levitan kernel

For simplicity we write

$$K(x, y) \equiv \langle x|K|y\rangle,$$
$$R(x, y) = F(x+y) + G(x-y) + \sum_i \Psi_{0i}(x)\Psi_{0i}^*(y)/C_i \tag{A1}$$

where $F(x)$ and $G(x)$ are given by equation (8). The Gelfand–Levitan equation (12) takes the form

$$K(x, y) = -R(x, y) - \operatorname{sgn} x \int_{-x}^{+x} K(x, z)R(z, y)\,\mathrm{d}z, \tag{A2}$$

where sgn x is the signum function

$$\operatorname{sgn} x = 1 \text{ if } x > 0 \qquad \operatorname{sgn} x = -1 \text{ if } x < 0.$$

We note that

$$\mathrm{d}/\mathrm{d}x(\operatorname{sgn} x) = 2\delta(x), \tag{A3}$$

and

$$\partial^2/\partial x^2 R(x, y) = \partial^2/\partial y^2 R(x, y). \tag{A4}$$

Equations (18) and (20) become

$$\Psi_k(x) = \mathrm{e}^{\mathrm{i}kx} + \operatorname{sgn} x \int_{-x}^{+x} K(x, y)\,\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}y,$$
$$\Psi_i(x) = \Psi_{0i}(x) + \operatorname{sgn} x \int_{-x}^{+x} K(x, y)\Psi_{0i}(y)\,\mathrm{d}y. \tag{A5}$$

We shall prove

$$(d^2/dx^2 + k^2)\Psi_k(x) = V\Psi_k(x), \tag{A6}$$

and the analogous expression for the point spectrum eigenfunctions, where V is the non-local potential given by equations (1) and (14), the latter equation being repeated in the new notation as

$$V_1(x) = 4\delta(x)K(0, 0) + 2 \operatorname{sgn} x \, dK(x, x)/dx,$$
$$V_2(x) = 2 \operatorname{sgn} x \, dK(x, -x)/dx. \tag{A7}$$

Now from the first of equations (A5)

$$d\Psi_k(x)/dx = ik\, e^{ikx} + \operatorname{sgn} x\,[K(x, x)\, e^{ikx} + K(x, -x)\, e^{-ikx}] + \operatorname{sgn} x \int_{-x}^{+x} K_x(x, y)\, e^{iky}\, dy. \tag{A8}$$

$$\begin{aligned} d^2\Psi_k(x)/dx^2 = {} & -k^2 e^{ikx} + 4\delta(x)K(0, 0)\, e^{ikx} \\ & + \operatorname{sgn} x\left(\frac{d}{dx}K(x, x)\cdot e^{ikx} + \frac{d}{dx}K(x, -x)\cdot e^{-ikx}\right) \\ & + ik \operatorname{sgn} x\,[K(x, x)\, e^{ikx} - K(x, -x)\, e^{-ikx}] \\ & + \operatorname{sgn} x\,[K_x(x, x)\, e^{ikx} + K_x(x, -x)\, e^{-ikx}] + \operatorname{sgn} x \int_{-x}^{+x} K_{xx}(x, y)\, e^{iky}\, dy. \end{aligned} \tag{A9}$$

In (A9) and later we use $K_x(x, y) = \partial K(x, y)/\partial x$ and $K_y(x, y) = \partial K(x, y)/\partial y$, and similarly for higher-order partial derivatives. Also

$$\begin{aligned} k^2\Psi_k(x) &= k^2 e^{ikx} + \operatorname{sgn} x \int_{-x}^{+x} K(x, y)k^2 e^{iky}\, dy \\ &= k^2 e^{ikx} - \operatorname{sgn} x \int_{-x}^{+x} K(x, y)\frac{d^2}{dy^2} e^{iky}\, dy. \end{aligned} \tag{A10}$$

We now integrate by parts twice to obtain

$$\begin{aligned} k^2\Psi_k(x) = {} & k^2 e^{ikx} - ik \operatorname{sgn} x\,[K(x, x)\, e^{ikx} - K(x, -x)\, e^{-ikx}] \\ & + \operatorname{sgn} x\,[K_y(x, x)\, e^{ikx} - K_y(x, -x)\, e^{-ikx}] - \operatorname{sgn} x \int_{-x}^{+x} K_{yy}(x, y)\, e^{iky}\, dy. \end{aligned} \tag{A11}$$

Hence

$$(d^2/dx^2 + k^2)\Psi_k(x) = V e^{ikx} + \int_{-x}^{+x} [K_{xx}(x, y) - K_{yy}(x, y)]\, e^{iky}\, dy. \tag{A12}$$

In obtaining V in equation (A12) we used $K_x(x, x) + K_y(x, x) = dK(x, x)/dx$ and $K_x(x, -x) - K_y(x, -x) = dK(x, -x)/dx$. If we can now show that

$$K_{xx}(x, y) - K_{yy}(x, y) = VK(x, y) \tag{A13}$$

where the non-local operator V acts only with respect to the x variable (i.e. $VF(x, y) = V_1(x)F(x, y) + V_2(x)F(-x, y)$ for any function ($F(x, y)$ of the two variables x and y), we shall have verified equation (A6) through the further use of the first of equations (A5).

We shall now sketch the proof of (A13), much of which follows the derivation of (A12). It will depend upon the uniqueness of the solutions of the Gelfand–Levitan equation (12) or equivalently (A12). In the case that the conditions of (7) and (7*a*) are satisfied, it is shown in Kay and Moses (1955, 1956a, b) that the solution is indeed unique.

We shall make use of the following lemma: if the solution of the Gelfand–Levitan equation is unique, then the only solution of the homogeneous linear integral equation

$$M(x, y) = -\operatorname{sgn} x \int_{-x}^{+x} M(x, z)\, R(z, y)\, \mathrm{d}z \tag{A14}$$

is

$$M(x, y) \equiv 0. \tag{A15}$$

Since the proof of the lemma follows the usual lines, we shall omit it.

From (A2)

$$\begin{aligned} K_{xx}(x, y) = &-R_{xx}(x, y) - 4\delta(x) K(0, 0) R(0, y) \\ &-\operatorname{sgn} x \left(\frac{\mathrm{d}}{\mathrm{d}x} K(x, x)\, R(x, y) + \frac{\mathrm{d}}{\mathrm{d}x} K(x, -x) R(-x, y) \right) \\ &-\operatorname{sgn} x\, [K(x, x) R_x(x, y) - K(x, -x) R_x(x, -x)] \\ &-\operatorname{sgn} x\, [K_x(x, x) R(x, y) + K_x(x, -x) R(x, -x)] \\ &+\operatorname{sgn} x \int_{-x}^{+x} K_{xx}(x, z) R(z, y)\, \mathrm{d}z. \end{aligned} \tag{A16}$$

But

$$K_{yy}(x, y) = -R_{yy}(x, y) - \operatorname{sgn} x \int_{-x}^{+x} K(x, z) R_{yy}(z, y)\, \mathrm{d}z. \tag{A17}$$

On using (A4)

$$K_{yy}(x, y) = -R_{xx}(x, y) - \operatorname{sgn} x \int_{-x}^{+x} K(x, z) R_{zz}(z, y)\, \mathrm{d}z. \tag{A18}$$

We can now integrate by parts twice in the integral of (A18) to obtain

$$\begin{aligned} K_{yy}(x, y) = &-R_{xx}(x, y) - \operatorname{sgn} x\, [K(x, x) R_x(x, y) - K(x, -x) R_x(x, -x)] \\ &+\operatorname{sgn} x\, [K_y(x, x) R(x, y) - K_y(x, -x) R(-x, y)] \\ &-\operatorname{sgn} x \int_{-x}^{+x} K_{zz}(x, z) R(z, y)\, \mathrm{d}z. \end{aligned} \tag{A19}$$

Then on defining $G(x, y)$ by

$$G(x, y) = K_{xx}(x, y) - K_{yy}(x, y) \tag{A20}$$

we obtain on subtracting (A19) from (A16)

$$G(x, y) = -VR(x, y) - \operatorname{sgn} x \int_{-x}^{+x} G(x, z) R(z, y)\, \mathrm{d}z. \tag{A21}$$

A solution of (A21) is

$$G(x, y) = VK(x, y) \tag{A22}$$

(see (A20).

Equation (A22) is the *only* solution of (A21), for if there were another solution, the homogeneous integral equation (A14) would have a non-trivial solution, which from the uniqueness of the solution of the Gelfand–Levitan equation (A2) is impossible. Thus we have proved equation (A13) and we have verified equation (14) for the continuous spectrum eigenfunctions. That the discrete spectrum eigenfunctions also satisfy the appropriate Schrödinger equation with the non-local potential is proved similarly.

It should be mentioned that the completeness relations (19) follow closely the proofs given in Abraham and Moses (1980) and DeFacio and Moses (1980) for local potentials.

References

Abraham P B and Moses H E 1980 *Phys. Rev.* A **22** 1333
Abraham P B, DeFacio B and Moses H E 1982 *Stud. Appl. Math.*
DeFacio B and Moses H E 1980 *J. Math. Phys.* **21** 1716
Faddeev L D 1976 *J. Sov. Math.* **5** 334
Gelfand I M and Levitan B M 1951 *Am. Math. Soc. Transl., Ser* 2 **1** 253
Kay I and Moses H E 1955 *Nuovo Cimento* **2** 917†
——1956a *Nuovo Cimento* **3** 66†
——1956b *Nuovo Cimento* **3** 276†
——1956c *J. Appl. Phys.* **27** 1503†
——1961a *Nuovo Cimento* **22** 689†
—— 1961b *Commun. Pure Appl. Math.* **14** 435
Moses H E 1975 *J. Math. Phys.* **16** 1044
——1979 *Mathematical Methods and Applications of Scattering Theory, Lecture Notes in Physics* Vol 130, ed J A DeSanto, A W Sáenz and W W Zachary (Berlin: Springer) p 260
Newton R G 1974 *The Three-Dimensional Inverse Scattering Problem in Quantum Mechanics, invited lecture delivered at the* 1974 *Summer Seminar on Inverse Problems, Am. Math. Soc. (August* 5–16, 1974)
——1980 *J. Math. Phys.* **21** 493

† Reprinted in Kay I and Moses H E 1982 *Inverse Scattering Papers: 1956–1963* (Math. Sci. Press)